Dynamical Entropy of Quantum Random Walks

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Overview

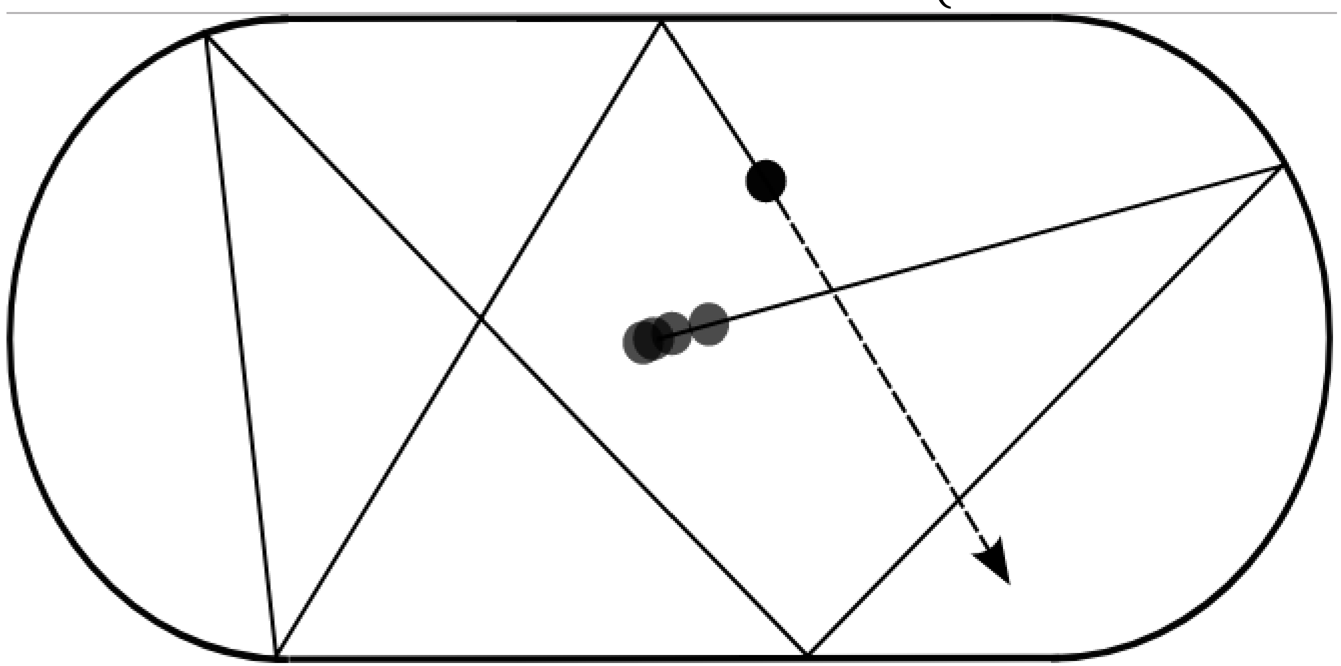
- Dynamical Systems
- Quantum Random Walks
- Entropy
- Applications of Entropy in Classical Information Theory
- Quantum Dynamical Entropy
- Applications in Quantum Information Theory

Classical Dynamical Systems

$(\Omega, \Sigma, \mu, f) =$ A probability space with a function describing the time dependence of points in that space.

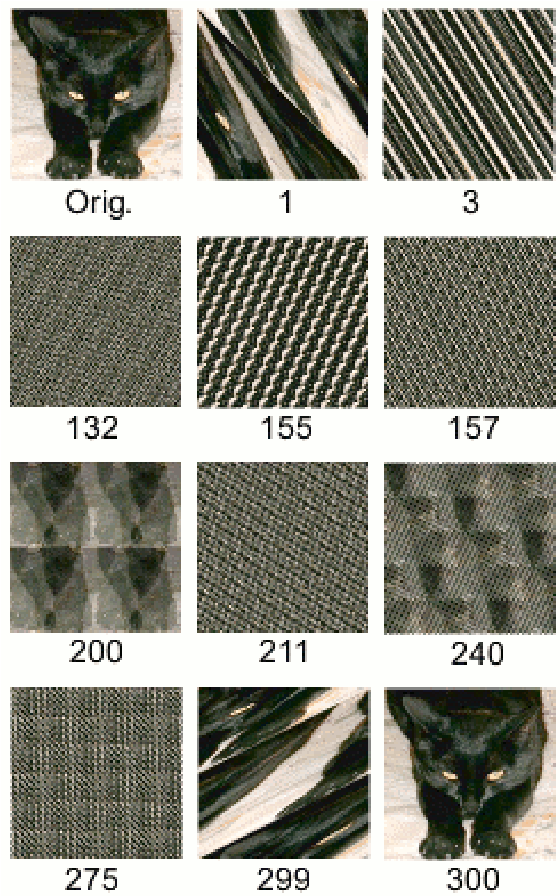
<u>Frictionless Billiards</u>

$$H(p,q) = \frac{p^2}{2m} + V(q), \text{ where } V(q) = \begin{cases} 0, & q \in \Omega \\ \infty, & q \notin \Omega \end{cases}$$



Source: https://en.wikipedia.org/wiki/Dynamical_billiards

Arnold Cat Map

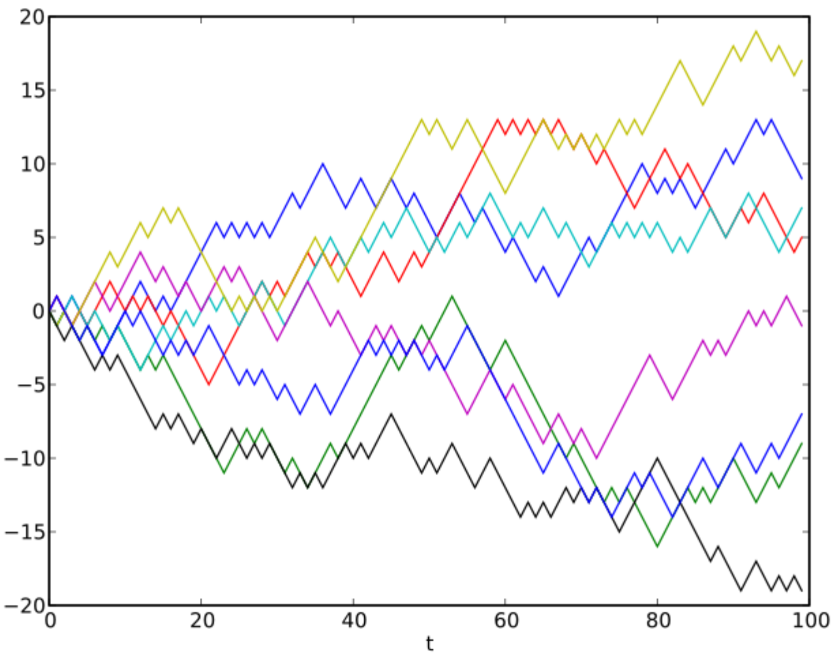


Source: By Claudio Rocchini - Own Work (It's not proper Arnold's cat but my black cat, due copyright restrictions), CC BY 2.5, https://commons.wikimedia.org/w/index.php?curid=1350710

Classical Random Walks as Dynamical Systems

<ins>Space:</ins> <ins>Transition Matrix:</ins> <ins>Initial State:</ins>

$$\mathbb{Z} \qquad P = \begin{bmatrix} \ddots & \ddots & \ddots & & & \\ \ddots & \frac{1}{2} & 0 & \frac{1}{2} & \ddots & \\ & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \qquad |0\rangle = \begin{pmatrix} \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$$



<ins>Space:</ins> <ins>Dynamics:</ins>

$\mathbb{Z}^{\mathbb{N}}$ $f(x) = y$, where $y_n = x_{n+1}$ for all $n \in \mathbb{N}$

Source: Wikipedia- Random Walk

Formalisms of Quantum Mechanics

Def: <u>Hilbert Space</u>

Complete, Inner Product Space

- Cauchy sequences converge
- Sesquilinear Map

$$\langle \cdot, \cdot \rangle : H^2 \to \mathbb{C}$$

1. $\langle x, y_1 \rangle = \overline{\langle y_1, x \rangle}$
2. $\langle ay_1 + y_2, x \rangle = a\langle y_1, x \rangle + \langle y_2, x \rangle$
3. $0 \leq \langle x, x \rangle = \|x\|^2$

Ex: $\mathbb{C}^n$

$$\langle x, y \rangle = \sum_{i=1}^{n} \overline{x_i} y_i$$

$$0 \leq \langle x, x \rangle = \sum_{i=1}^{n} |x_i|^2$$

Formalisms of Quantum Mechanics

<u>Hilbert Space</u>

Def: <u>Linear Functionals</u>

$$\langle y| : H \to \mathbb{C}$$

"Bra" "Ket"

$$\langle y| \qquad |x\rangle$$

$$\Cap \qquad \Cap$$

$$H^* = H$$

$$\langle y||x\rangle = \langle y, x\rangle$$

Ex: $|x\rangle \in \mathbb{C}^n$

$$\langle y| \in \mathbb{C}^n$$

Formalisms of Quantum Mechanics

<u>Hilbert Space</u>

<u>Linear Functionals</u>

Def: <u>Pure</u> <u>State</u>

$$T : H \to H \qquad tr(T) = 1$$

$$|x\rangle\langle x| \qquad tr(|x\rangle\langle x|) = \langle x, x\rangle$$

$$|x\rangle\langle x| = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \begin{pmatrix} \overline{x_1} & \overline{x_2} & \cdots & \overline{x_n} \end{pmatrix} = \begin{pmatrix} |x_1|^2 & \cdots & x_1\overline{x_n} \\ \vdots & \ddots & \vdots \\ x_n\overline{x_1} & \cdots & |x_n|^2 \end{pmatrix}$$

Ex: $|x\rangle \in \mathbb{C}^n$

$\langle y| \in \mathbb{C}^n$

$|x\rangle\langle x| \in M_n(\mathbb{C})$

From Classical to Quantum

Space: $\mathbb{Z}$ $\longrightarrow$ $\ell_2(\mathbb{Z})$

Probability Distribution

Pure State = $|x\rangle\langle x|$

State: $\sum p_i = 1$ $\longrightarrow$ $\sum |x_i|^2 = 1$

Evolution: Transition Matrix $\longrightarrow$ Positive, Trace-Preserving Operators

Evolution of a Quantum System

$$\Theta(\rho) = \sum_k A_k \rho A_k^*$$

where $$\sum_k A_k^* A_k = \mathbb{1}$$

In particular,

$$\Theta(|x\rangle\langle x|) = U|x\rangle\langle x|U^* = |Ux\rangle\langle Ux|$$

Quantum Random Walk

Internal Degrees of Freedom: $H_C = \mathbb{C}^2$ with basis $\{|\uparrow\rangle, |\downarrow\rangle\}$

Position Space: $H_P = \ell_2(\mathbb{Z})$

Where the Magic happens: $H = \mathbb{C}^2 \otimes \ell_2(\mathbb{Z})$

with basis elements $|\downarrow, n\rangle$ and $|\uparrow, n\rangle$

Quantum Random Walk

Coin Space:

$$h = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$h|\uparrow\rangle = \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}$$

Gives equal probability to be in spin up or spin down.

$$h \otimes \mathbb{1}_P = \frac{1}{\sqrt{2}}\left(\begin{array}{ccc|ccc} \ddots & 0 & 0 & \ddots & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & \ddots & 0 & 0 & \ddots \\ \hline \ddots & 0 & 0 & \ddots & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & \ddots & 0 & 0 & \ddots \end{array}\right)$$

Quantum Random Walk

Shift Operator:

$$S = \sum_{n \in \mathbb{Z}} |\uparrow, n+1\rangle\langle\uparrow, n| + |\downarrow, n-1\rangle\langle\downarrow, n|$$

If particle is in spin up, S will shift it right.
If particle is in spin down, S will shift it left.

$$S = \left(\begin{array}{ccc|ccc} \ddots & 0 & 0 & \ddots & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 & \ddots \\ \hline \ddots & 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \ddots & 0 & 0 & \ddots \end{array}\right)$$

Quantum Random Walk

Unitary Operator:

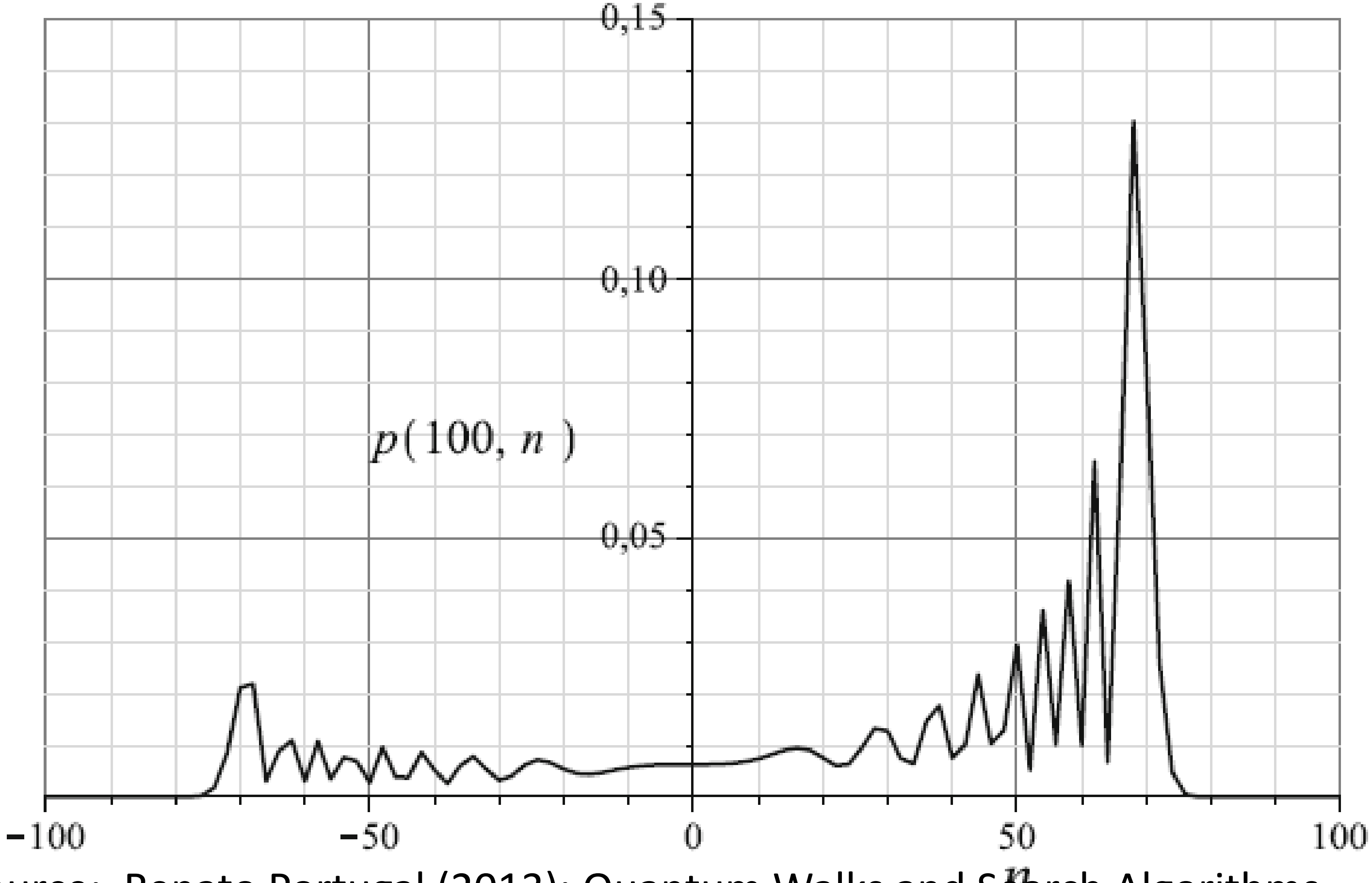
$$U = S(h \otimes \mathbb{1}_p) = \frac{1}{\sqrt{2}} \left(\begin{array}{ccc|ccc} \ddots & 0 & 0 & \ddots & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & \ddots & 0 & 1 & \ddots \\ \hline \ddots & 1 & 0 & \ddots & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & \ddots & 0 & 0 & \ddots \end{array}\right)$$

Now we have options for our initial state even after restricting it to be at the origin.

$$|\uparrow, 0\rangle = \left(\begin{array}{c} \vdots \\ 1 \\ \vdots \\ \hline \vdots \\ 0 \\ \vdots \end{array}\right) \quad \text{or} \quad |\downarrow, 0\rangle = \left(\begin{array}{c} \vdots \\ 0 \\ \vdots \\ \hline \vdots \\ 1 \\ \vdots \end{array}\right)$$

Quantum Random Walk

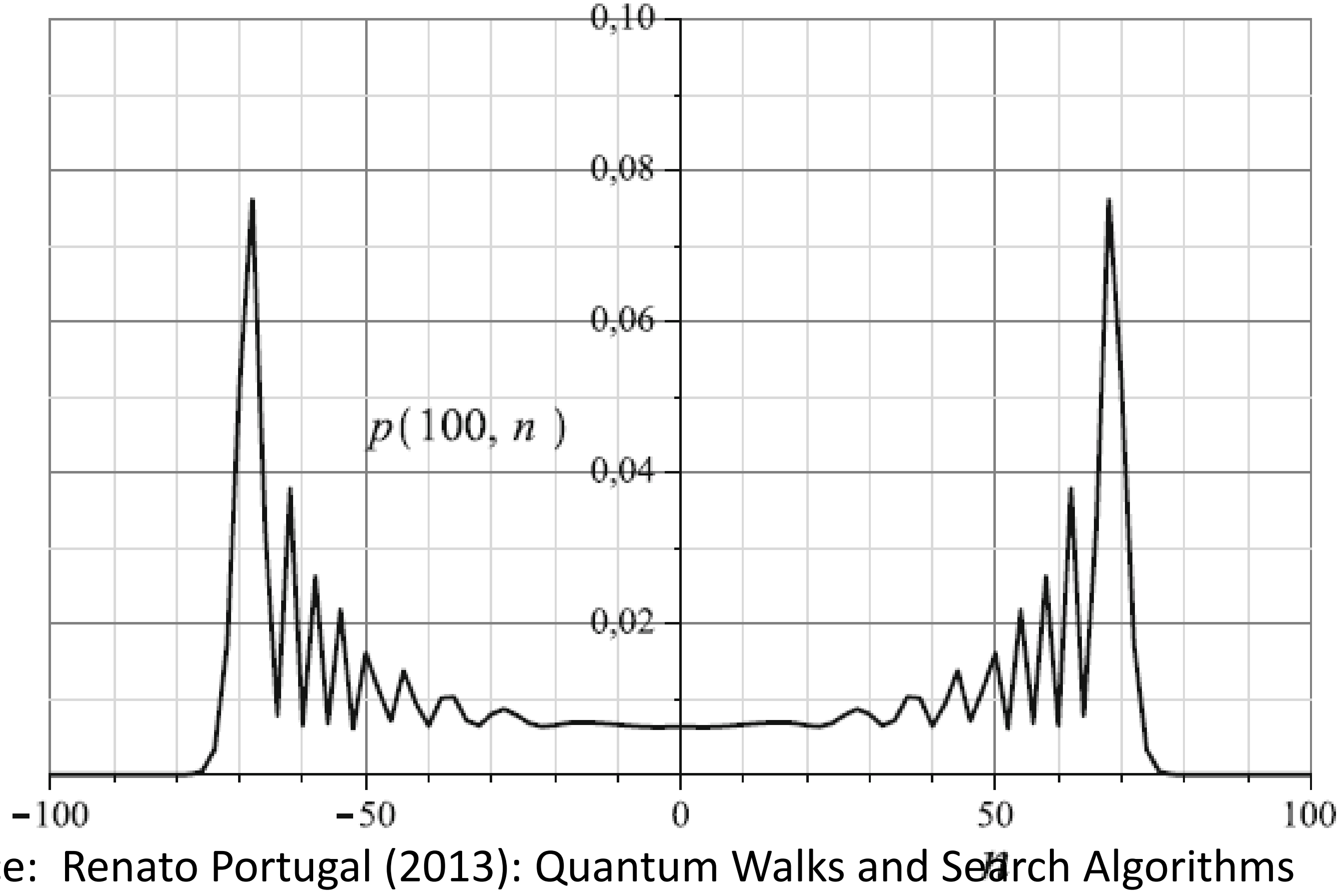
Initial State: $|\uparrow, 0\rangle$



Source: Renato Portugal (2013): Quantum Walks and Search Algorithms

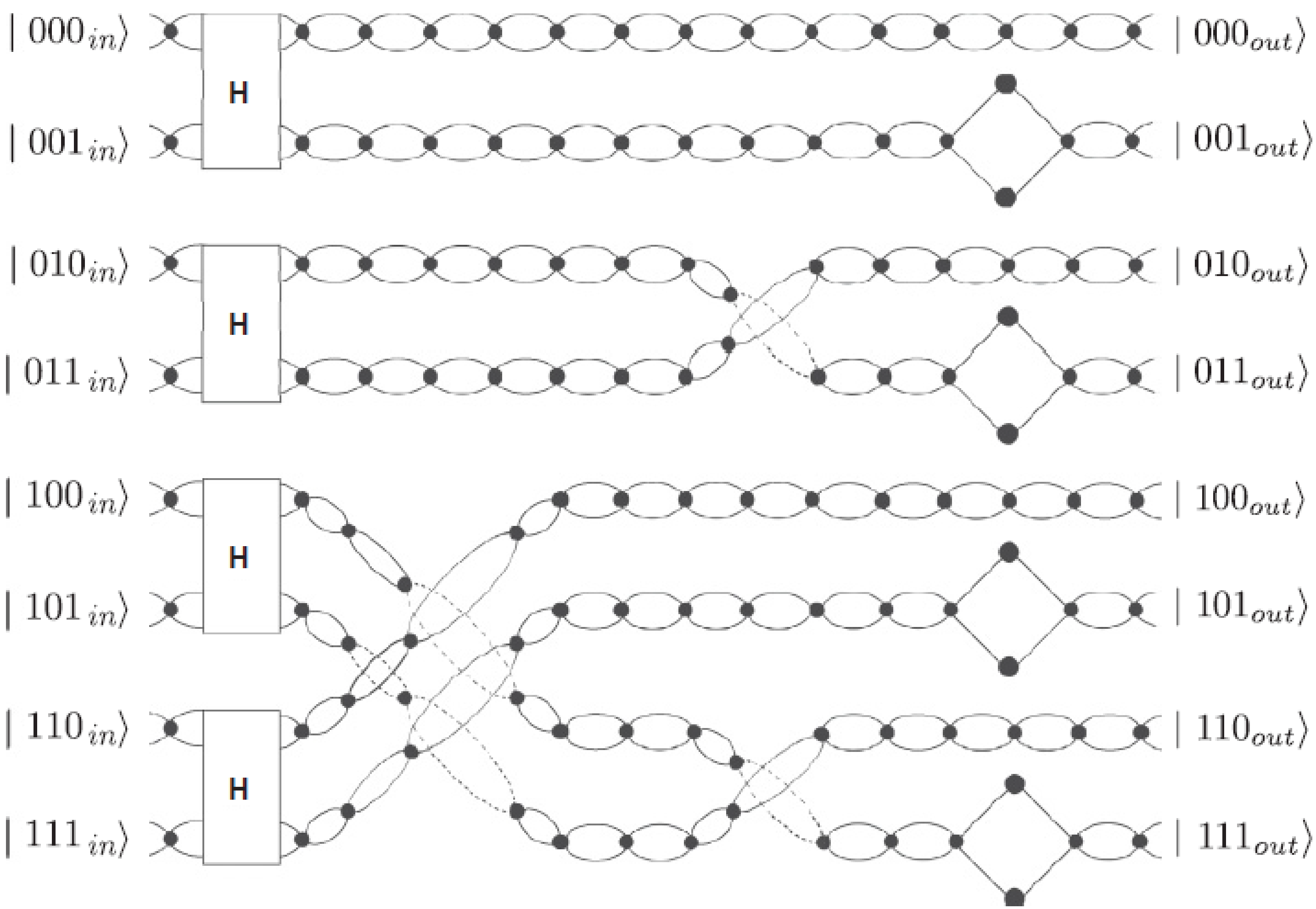
Quantum Random Walk

Initial State: $\dfrac{|\uparrow, 0\rangle + i|\downarrow, 0\rangle}{\sqrt{2}}$



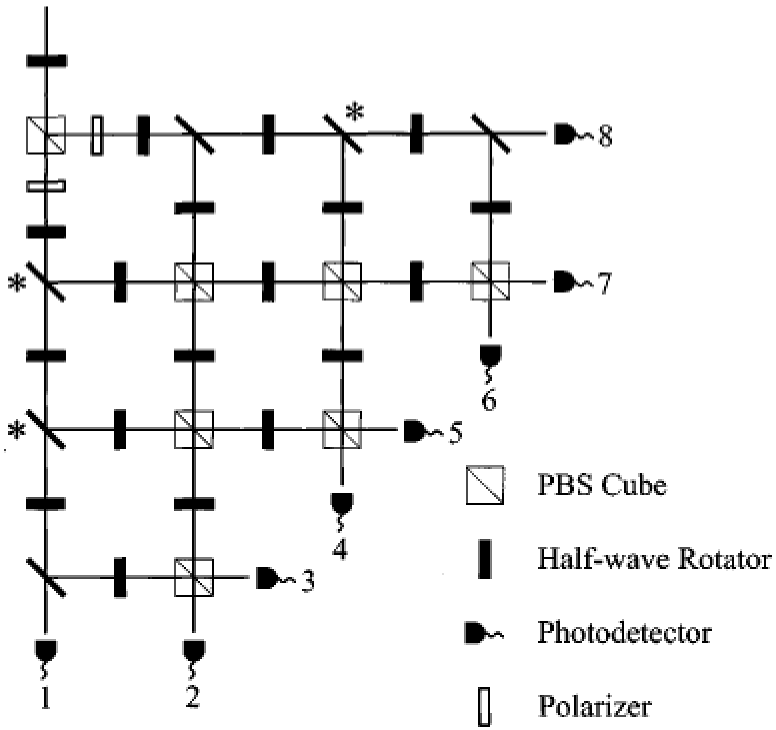
Source: Renato Portugal (2013): Quantum Walks and Search Algorithms

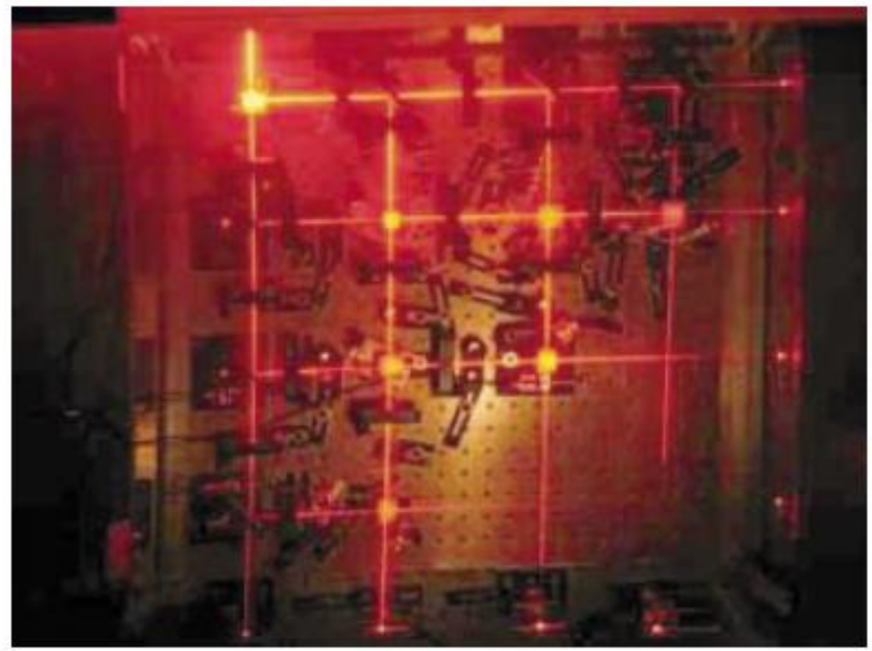
Universal for Quantum Computation



Source: *Universal quantum computation using the discrete-time quantum walk*, Lovett et. al.

Implementation in Linear Optics





Source: *Experimental realization of a quantum quincunx by use of linear optical elements*, Do et. al.

Entropy

We have a classical system whose macrostate is described by the probability measure

$$p = (p_1, p_2, \dots, p_k)\ .$$

After measuring the system N times, we expect to see:

- 1st microstate: $p_1 N$ times
- 2nd microstate: $p_2 N$ times
- ⋮
- kth microstate: $p_k N$ times

$$\frac{1}{N}\log\frac{N!}{(p_1N)!\,(p_2N)!\cdots(p_kN)!}\ \xrightarrow[N\ \to\ \infty]{}\ -\sum_{i=1}^{k} p_i \log p_i$$

$$H(X) = -\sum_{i=1}^{k} p_i \log p_i = \sum_{i=1}^{k} \eta(p_i)$$

Entropy Rate

<u>Stochastic Process:</u> $\boldsymbol{X} = (X_n)_{n=1}^{\infty}$

<u>Entropy Rate:</u> $H(\boldsymbol{X}) = \lim_{n\to\infty} \frac{1}{n} H(X_1, X_2, \ldots, X_n)$
$= \lim_{n\to\infty} \sum_{i_1,i_2,\ldots,i_n}^{k} \eta(p_{i_1,i_2,\ldots,i_n})$

<u>Markov Process:</u> $H(\boldsymbol{X}) = \lim_{n\to\infty} \frac{1}{n} H(X_1, X_2, \ldots, X_n)$
$= \sum_{i=1}^{k} p_i \sum_{j=1}^{k} \eta(p_{j|i})$,
where $p = (p_1, p_2, \ldots, p_k)$ is an invariant measure.

<u>Unbiased Random Walk:</u> $H = \sum_{i=1}^{k} \frac{1}{k} \sum_{j=1}^{k} \eta(p_{j|i})$
$= \log 2$

SZ Quantum Dynamical Entropy

Dynamical System: (Schrödinger Picture)

$(\Theta, \mathrm{T}, \rho)$ where $\Theta(\cdot) = U \cdot U^*, \rho \in S_1(H)$ and $\mathrm{T}(A) := \sum_{i \in A} P_i \cdot P_i$.

Probabilities: $p_{i_1, i_2, \ldots, i_n} = \mathrm{tr}(\mathrm{T}(i_n) \circ \Theta \circ \mathrm{T}(i_{n-1}) \circ \cdots \circ \Theta \circ \mathrm{T}(i_1)\rho)$

SZ Dynamical Entropy: $h^{SZ}(\Theta, \mathrm{T}, \rho) = \mathrm{limsup}_{n \to \infty} \frac{1}{\mathrm{n}} \sum_{i \in \Omega} \eta(p_{i_1, i_2, \ldots, i_n})$

Theorem 1. (Androulakis, Wright)

Let Θ = Hadamard walk on N-cycle $\mathrm{H_C} \otimes H_P = \mathbb{C}^2 \otimes \mathbb{C}^N$,
and $\mathrm{T} = (P_n)_{n=1}^N$ with $P_n = \mathbb{1}_C \otimes |n\rangle\langle n|$, and $\rho = \mathbb{1}/2N$.

Then $h^{SZ}(\Theta, \mathrm{T}, \rho) = \log 2$

and $h^{SZ}(\Theta^2, \mathrm{T}, \rho) = \dfrac{4}{3} \log 2.$

$\Longrightarrow$

Nonlinear in time: In classical dynamical entropy we have

$$n h^{KS}(f) = h^{KS}(f^n).$$

AOW Quantum Dynamical Entropy

Dynamical System: (Heisenberg Picture)

$(\mathcal{A}, \Theta^*, \phi)$ where $\Theta^*(\cdot) = U^* \cdot U$ and $\phi \in S(\mathcal{A})$.

Quantum Markov Chains:

$\gamma = (P_i)_{i=1}^d$, $\mathbb{E}: M_d \otimes \mathcal{A} \to \mathcal{A}$ defined by $\mathbb{E}(\sum_{i,j=1}^d |i\rangle\langle j| A_{i,j}) = \Theta^*(\sum_{i=1}^d P_i A_{i,i} P_i)$

The Markov state $\phi_\infty \in S(M_d^{\otimes \mathbb{N}})$ is given by

$$\phi_\infty(a_1 a_2 \cdots a_n) = \phi(\mathbb{E}\left(a_1 \otimes \mathbb{E}(a_2 \otimes \cdots \mathbb{E}(a_{n-1} \otimes \mathbb{E}(a_n \otimes 1_{\mathcal{A}}) \cdots))\right))$$

Let $\rho_n \in M_d^{\otimes n}$ satisfy $\phi_\infty(a_1 a_2 \cdots a_n) = \mathrm{tr}\big(\rho_n \mathbb{E}(a_1 \otimes \cdots \mathbb{E}(a_n \otimes 1_{\mathcal{A}}) \cdots)\big)$

AOW Dynamical Entropy: $h^{AOW}(\Theta^*, \gamma, \phi) = \mathrm{limsup}_{n\to\infty} \frac{1}{n} \mathrm{S}(\rho_n)$

where $S(\rho) = \mathrm{tr}(\eta(\rho))$ is the von Neumann entropy.

SZ=AOW Dynamical Entropy

Theorem 2. (Androulakis, Wright)

Given a dynamical system

$$(\Theta, \mathrm{T}, \rho) \qquad \text{or} \qquad (\mathcal{A}, \Theta^*, \phi),$$

$$h^{SZ}(\Theta, \mathrm{T}, \rho) \;=\; h^{AOW}(\Theta^*, \gamma, \phi).$$

Proof.

$$\begin{aligned}
p_{i_1,i_2,\dots,i_n} &= \mathrm{tr}(\mathrm{T}(i_n) \circ \Theta \circ \mathrm{T}(i_{n-1}) \circ \cdots \circ \Theta \circ \mathrm{T}(i_1)\rho) \\
&= \mathrm{tr}\Big(\mathrm{T}(i_{n-1}) \circ \Theta \circ \mathrm{T}(i_{n-2}) \circ \cdots \circ \Theta \circ \mathrm{T}(i_1)\rho\mathbb{E}\big(E_{i_n,i_n} \otimes 1_{\mathcal{A}}\big)\Big) \\
&\;\vdots \\
&= \mathrm{tr}\Big(\mathrm{T}(i_1)\rho\mathbb{E}(E_{i_2,i_2} \otimes \mathbb{E}\Big(\cdots \mathbb{E}\big(E_{i_n,i_n} \otimes 1_{\mathcal{A}}\big)\Big))\Big) \\
&= \mathrm{tr}\Big(\rho\mathbb{E}(E_{i_1,i_1} \otimes \mathbb{E}(E_{i_2,i_2} \otimes \mathbb{E}\Big(\cdots \mathbb{E}\big(E_{i_n,i_n} \otimes 1_{\mathcal{A}}\big)\Big)\Big))\Big) \\
&= \rho_n(i_1, i_2, \dots, i_n; i_1, i_2, \dots, i_n)
\end{aligned}$$

Compressability of Data

$$\text{OBJECTS} = S \xrightarrow{\quad C \quad} \text{CODEWORDS} \subset A^+ = \cup_{\ell=0}^{\infty} \{0,1\}^n$$

The <u>Source Code</u> *C* is uniquely decodable if its extension $C^+ : S^+ \to A^+$

$$C^+(x_1 x_2 \cdots x_n) = C(x_1)C(x_2)\cdots C(x_n)$$

is one-to-one, for all *n*.

<u>Kraft-McMillan Inequality.</u>

Any uniquely decodable code with codeword lengths $\ell_1, \ell_2, \ldots, \ell_n$ must satisfy the inequality $\sum_{i=1}^{n} 2^{-\ell_i} \leq 1.$
Conversely, given lengths that satisfy the above inequality there exists a uniquely decodable code with those lengths.

Optimal Lossless Codes

Shannon's Noiseless Coding Theorem.

Given a random variable X, the optimal source code C satisfies the inequality

$$H(X) \leq L(C) < H(X) + 1,$$

where $L(C) = \mathbb{E}[\ell(x)] = \sum_{x \in S} p(x)\ell(x)$ is the expected length of C.

Corollary.

Given a stochastic process $\boldsymbol{X} = (X_n)_{n=1}^{\infty}$, the optimal source code C_n for the strings of length n satisfies the inequality

$$H(X_1, X_2, \ldots, X_n) \leq L(C_n) < H(X_1, X_2, \ldots, X_n) + 1.$$

Therefore average expected length per symbol $L_n^* = \frac{1}{n} L(C_n)$ is given by

$$H(\boldsymbol{X}) = \lim_{n \to \infty} \frac{1}{n} H(X_1, X_2, \ldots, X_n) = \lim_{n \to \infty} L_n^* =: L^*.$$

In particular, if ***X*** has i.i.d. copies of a random variable *X*, then

$$H(\boldsymbol{X}) = L^* = H(X).$$

Compressing Quantum Data

$$\text{OBJECTS} \;=\; \mathcal{S} \subset H_{\mathcal{S}} \xrightarrow{\quad U \quad} \text{CODEWORDS} \;\subset\; H_{\mathcal{A}}^{\oplus} \;=\; \bigoplus_{\ell=0}^{\infty} H_{\mathcal{A}}^{\otimes n}$$

where $\mathcal{S} = \{p_n, |s_n\rangle\}_{n=1}^{N}$ is an ensemble of states in $H_{\mathcal{S}} = \text{span}\{|s_n\rangle\} = \mathbb{C}^d$ and $H_{\mathcal{A}} = \mathbb{C}^2 = \text{span}\{|0\rangle, |1\rangle\}$.

The Quantum Source Code U is uniquely decodable if its extension $U^+ : H_{\mathcal{S}}^{\oplus} \to H_{\mathcal{A}}^{\oplus}$

$$U^+(x_1 x_2 \cdots x_n) = U(x_1)U(x_2)\cdots U(x_n)$$

is a linear isometry, for all n.

We define the length observable $\Lambda \in B(H_{\mathcal{A}}^{\oplus})$ by

$$\Lambda \;:=\; \sum_{\ell=0}^{\ell_{\max}} \ell \Pi_{\ell}$$

where Π_{ℓ} is the orthogonal projection onto the subspace $H_{\mathcal{A}}^{\otimes \ell} \subset H_{\mathcal{A}}^{\oplus}$.
The quantum codeword length of $|\omega\rangle \equiv U|s\rangle$ for each $|s\rangle \in H_{\mathcal{S}}$ is given by

$$\ell(|\omega\rangle) \equiv \langle\omega|\Lambda|\omega\rangle .$$

Quantum from Classical

Let $C: S \to A^+$ be a classical uniquely decodable code with $|S| = \dim(H_{\mathcal{S}})$. Then for any orthonormal basis $\{|e_i\rangle\}_{i=1}^{d}$ of $H_{\mathcal{S}}$,

$$U = \sum_{i=1}^{d} |C(x_i)\rangle\langle e_i|$$

is uniquely decodable. Furthermore, the quantum codeword lengths for $|\omega\rangle \equiv U\,|s\rangle$ are given by

$$\ell(|\omega\rangle) \equiv \langle\omega|\Lambda|\omega\rangle = \sum_{i=1}^{d} |\langle e_i|s\rangle|^2\,\ell_i.$$

<u>Theorem 3</u>. (Quantum Kraft-McMillan Inequality)

Any uniquely decodable code U must satisfy the inequality

$$\mathrm{tr}(U^\dagger 2^{-\Lambda} U) \leq 1.$$

Conversely, if $U: H_{\mathcal{S}} \to H_{\mathcal{A}}^{\oplus}$ is a linear isometry with length eigenstates satisfying the above inequality, then there exists a uniquely decodable quantum code (of the above form) with the same number of length ℓ eigenstates, for each $\ell \in \mathbb{N}$.

Optimal Quantum Lossless Codes

Let $\mathcal{S} = \{p_n, |s_n\rangle\}_{n=1}^{N}$ and $\rho = \sum_{n=1}^{N} p_n |s_n\rangle\langle s_n|$.

Suppose ρ has spectral decomposition

$$\rho = \sum_{i=1}^{d} \rho_i |\rho_i\rangle\langle\rho_i|.$$

Theorem 4. (Bellomo, Bosyk, Holik, Zozor 2017)

The optimal classical-quantum source code is given by

$$U = \sum_{i=1}^{d} |c(i)\rangle\langle\rho_i|$$

where $\{c(i)\}$ is the classical Huffman code for the probabilities $\{\rho_i\}$.

Optimal Quantum Lossless Codes

Theorem 5. (Bellomo, Bosyk, Holik, Zozor 2017)

The average length of the optimal quantum source code satisfies the inequalities

$$S(\rho) \;\leq\; \ell\big(\Gamma(\rho)\big) \;<\; S(\rho)+1,$$

$\Gamma(\cdot) = U \cdot U^{\dagger}$ and $\ell\big(\Gamma(\rho)\big) = \mathrm{tr}(\Gamma(\rho)\Lambda)$.

Corollary.

The average length of the optimal quantum source code for the i.i.d. ensemble $\mathcal{S}^{\otimes n}$ satisfies the inequalities

$$nS(\rho) = S\big(\rho^{\otimes n}\big) \;\leq\; \ell\left(\Gamma_{\mathrm{n}}\big(\rho^{\otimes n}\big)\right) \;<\; S\big(\rho^{\otimes n}\big)+1 = nS(\rho)+1.$$

Therefore $\lim_{n\to\infty}\frac{1}{n}\ell\left(\Gamma_n\big(\rho^{\otimes n}\big)\right) = S(\rho) = h^{AOW}(\Theta^*,\gamma,\phi)$, where $\gamma = (\,|\rho_i\rangle\langle\rho_i|\,)_{i=1}^{d}$,
Θ^* is the Bernoulli shift on $M_d^{\otimes\mathbb{N}}$ and $\phi(a_1a_2\cdots a_n) = \mathrm{tr}\left(\rho^{\otimes n}\mathbb{E}(a_1\otimes\cdots\mathbb{E}(a_n\otimes 1_{\mathcal{A}})\cdots)\right)$.

Open Question. Can the above result relating the average length per symbol be extended to include a stochastic ensemble $\mathcal{S}^k = \{p_{n_1,\ldots,n_k}, |s_1s_2\cdots s_k\rangle\}_{n_1,\ldots,n_k=1}^{N}$?

Optical Communication Process

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\text{ALICE} & \text{BOB} \\[-2em]
\rho \in S_1(H) \arrow[r, "\Lambda"] \arrow[d, "\alpha"'] & S_1(H) \\
S_1(H \otimes K) \arrow[r, "\Theta"'] & S_1(H \otimes K) \arrow[u, "\beta"']
\end{tikzcd}
$$

Where

$$\alpha(\rho) = \rho \otimes \nu$$

for some noise ν coming from the noisy channel and

$$\beta(\varphi) = \mathrm{tr}_K(\varphi).$$

Optical Communication Process

<u>Example.</u>

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\text{ALICE} & & \text{BOB} \\
\rho \in S_1(H_P) \arrow[rr, "\Lambda"] \arrow[d, "\alpha"'] & & S_1(H_P) \\
S_1(H_P \otimes H_C) \arrow[rr, "\Theta"'] & & S_1(H_P \otimes H_C) \arrow[u, "\beta"']
\end{tikzcd}$$

Where $\Theta(\cdot) = U \cdot U^{\dagger}$ is the Hadamard walk on the N-cycle given by the unitary $U = S(1_P \otimes h)$, $\alpha(\rho) = \rho \otimes \nu$ where h$\nu = \nu$, and $\beta(\varphi) = \mathrm{tr}_{H_C}(\varphi)$.
Letting $\rho = 1_P/N$, $\mathrm{T}_1 = (P_n)_{n=1}^{N}$ where $P_n = |n\rangle\langle n|$, and $\mathrm{T}_2 = (Q_n)_{n=1}^{N}$ where $Q_n = |n\rangle\langle n| \otimes 1_C$, we find that

$$h^{SZ}(\Lambda, \mathrm{T}_1, \rho) \neq h^{SZ}(\Theta, \mathrm{T}_2, \rho \otimes \nu).$$

Thank you!